Commun. Combin., Cryptogr. & Computer Sci., 2 (2021), 179–185

ISSN:2783-5456

Communications in Combinatorics, Cryptography & Computer Science

Journal Homepage: http://cccs.sgh.ac.ir

First KCD Matrix and its Laplacian energy

Keerthi G. Mirajkar[a,*], Akshata Morajkar[b]

[a] *Department of Mathematics, Karnatak University's Karnatak Arts College, Dharwad - 580001, Karnataka, India.*

[b] *Department of Mathematics, Karnatak University's Karnatak Arts College, Dharwad - 580001, Karnataka, India.*

Abstract

This article introduces the concept associated with Laplacian energy for first KCD matrix named as first KCD Laplacian energy. Further, we investigate some basic results related to this concept and also develop significant bounds for first KCD Laplacian energy.

Keywords: First KCD Laplacian matrix, first KCD Laplacian eigenvalues, first KCD Laplacian energy.

2020 MSC: 05C07, 05C50.

1. Introduction

The graph G considered in this article is connected, simple and undirected, having $|V(G)| = n$ and $|E(G)| = m$. The vertex degree of v_i is represented as d_i. The graph G is regular if each vertex is of same degree. For undefined terminologies refer [5].

In 1978 graph energy concept was brought forward by Gutman [3] . It defines energy of G as the sum of absolute eigenvalues of G. Tremendous work on this concept is available in the literature [1, 9]. Recently various graph-energy-like quantities: Laplacian [4], distance [6] and others are studied.

The first Karnatak College Dharwad matrix i.e., first KCD matrix $KCD_1(G) = [kcd_{1_{ij}}]$ is defined in [7] as follows

$$kcd_{1_{ij}} = \begin{cases} (d_i + d_j) + d_e & \text{if } v_i \text{ is adjacent to } v_j \text{ ,} \\ 0 & \text{otherwise.} \end{cases}$$

with d_i and d_j representing degree of vertex v_i and v_j respectively, d_e is the edge degree given by $d_e = d_i + d_j - 2$. It is has order $n \times n$.

The first KCD eigenvalues [7] of G are $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_n$ and the corresponding first KCD energy $E_{KCD_1}(G)$ [7] is

$$E_{KCD_1}(G) = \sum_{i=1}^{n} |\beta_i|. \tag{1.1}$$

*Keerthi G. Mirajkar

Email addresses: keerthi.mirajkar@gmail.com (Keerthi G. Mirajkar), akmorajkar@gmail.com (Akshata Morajkar)

Received: March 11, 2022 Revised: May 23, 2022 Accepted: September 17, 2022

If $D(G)$ and $A(G)$ are the diagonal matrix, adjacency matrix respectively, then the Laplacian matrix [4] of G is $L(G) = D(G) - A(G)$. with Laplacian eigenvalues being labeled as $\mu_1 \geqslant \mu_2 \geqslant ... \geqslant \mu_n$.
The Lapalcian energy [4] of G is

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

The concept of Laplacian energy has numerous chemical applications. Various properties for this concept are studied in [8, 9]. In consideration with the Laplacian matrix and laplacian energy of G, we define the first KCD Laplacian matrix and first KCD Laplacian energy of G.

The Cauchy-Schwarz inequality [2] states, if $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$are n real vectors, then

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leqslant \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right) \tag{1.2}$$

In the second segment, we introduce the first KCD Laplacian matrix and first KCD Laplacian energy of G, followed by general basic results on them. Further, third segment establishes few bounds for the first KCD Laplacian energy.

2. First KCD Laplacian energy

Definition 2.1. Let $D(G)$ is the diagonal matrix and $KCD_1(G)$ is the first KCD matrix, then for a (n, m) graph G, the first Karnatak College Dharwad Laplacian matrix $L_{KCD_1}(G)$ is defined as

$$L_{KCD_1}(G) = [z_{ij}] = D(G) - KCD_1(G).$$

It has first KCD Laplacian eigenvalues $\eta_1 \geqslant \eta_2 \geqslant ... \geqslant \eta_n$ in the non-increasing order, where η_1 and η_n are the highest and lowest first KCD Laplacian eigenvalues of G.
The corresponding first Karnatak College Dharwad Laplacian energy $LE_{KCD_1}(G)$ is defined as

$$LE_{KCD_1}(G) = \sum_{i=1}^{n} |\xi_i| \tag{2.1}$$

where $\xi_i = \eta_i - \overline{d}$ with $\overline{d} = \dfrac{2m}{n}, 1 \leqslant i \leqslant n$.

The following lemma are used for calculating bounds of first KCD Laplacian energy.

Lemma 2.2. *For a regular graph* G,

$$1.\ \sum_{i=1}^{n} \eta_i = 2m \tag{2.2}$$

$$2.\ \sum_{i=1}^{n} \eta_i^2 = \sum_{i=1}^{n} \beta_i^2 + \sum_{i=1}^{n} d_i^2. \tag{2.3}$$

Proof. Consider,

$$\begin{aligned} \sum_{i=1}^{n} \eta_i &= trace\,(L_{KCD_1}(G)) \\ &= \sum_{i=1}^{n} d_i \\ &= 2m. \end{aligned}$$

Now,

$$\begin{aligned}\sum_{i=1}^{n}\eta_i^2 &= \operatorname{trace}\left(L_{KCD_1}(G)\right)^2\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} z_{ij}z_{ji}\\ &= 2\sum_{i<j} z_{ij}^2+\sum_{i=1}^{n} z_{ii}^2\\ &= 2\sum_{i<j}(d_i+d_j+d_e)^2+\sum_{i=1}^{n} d_i^2\\ &= \sum_{i=1}^{n}\beta_i^2+\sum_{i=1}^{n} d_i^2\end{aligned}$$

□

Lemma 2.3. *For a regular graph* G*,*

$$\text{i)}\quad \sum_{i=1}^{n}\xi_i = 0 \tag{2.4}$$

$$\text{ii)}\quad \sum_{i=1}^{n}\xi_i^2 = B,\ \ \text{where}\ \ B=\sum_{i=1}^{n}\beta_i^2+\sum_{i=1}^{n}\left(d_i-\overline{d}\right)^2 . \tag{2.5}$$

Proof. Consider,

$$\sum_{i=1}^{n}\xi_i = \sum_{i=1}^{n}(\eta_i-\overline{d})$$

Using Eq (2.2), we have

$$\begin{aligned}\sum_{i=1}^{n}\xi_i &= \sum_{i=1}^{n}\eta_i-2m\\ &= 0.\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\xi_i^2 &= \sum_{i=1}^{n}\left(\eta_i-\overline{d}\right)^2\\ &= \sum_{i=1}^{n}\eta_i^2-\frac{4m}{n}\sum_{i=1}^{n}\eta_i+\frac{4m^2}{n}\end{aligned}$$

Using Eq. (2.3), we get

$$\begin{aligned}\sum_{i=1}^{n}\xi_i^2 &= \sum_{i=1}^{n}\beta_i^2+\sum_{i=1}^{n}d_i^2-\frac{4m}{n}\sum_{i=1}^{n}\eta_i+\frac{4m^2}{n}\\ &= \sum_{i=1}^{n}\beta_i^2+\sum_{i=1}^{n}\left(d_i-\overline{d}\right)^2\\ &= B,\ \ \text{where}\ \ B=\sum_{i=1}^{n}\beta_i^2+\sum_{i=1}^{n}\left(d_i-\overline{d}\right)^2.\end{aligned}$$

□

Lemma 2.4. *For a regular graph* G,

$$LE_{KCD_1}(G) = E_{KCD_1}(G).$$

Proof. The regular graph G satisfies

$$\eta_i - \overline{d} = -\beta_{n-i+1}, \quad 1 \leqslant i \leqslant n \tag{2.6}$$

Substituting Eq. (2.6) in Eq. (2.1), generates

$$\begin{aligned} LE_{KCD_1}(G) &= \sum_{i=1}^{n} \left|\overline{d} - \beta_{n-i+1} - \overline{d}\right| \\ &= \sum_{i=1}^{n} |-\beta_{n-i+1}| \\ &= E_{KCD_1}(G). \end{aligned}$$

□

Remark 2.5. If G is a regular graph, then $\eta_n = 3r - 4r^2$.

Remark 2.6. For a graph G, $\eta_n < \overline{d}$.

3. Bounds for the first KCD Laplacian energy

Theorem 3.1. *For a regular graph* G,

$$\xi_1 < \sqrt{\frac{B(n-1)}{n}}.$$

Proof. Let $a_i = 1$ and $b_i = \xi_i$ for each $i = 2, 3, ..., n$ in inequality (1.2), we have

$$\left(\sum_{i=2}^{n} \xi_i\right)^2 < (n-1)\sum_{i=2}^{n} \xi_i^2 \tag{3.1}$$

Using Eq. (2.4),

$$\left(\sum_{i=2}^{n} \xi_i\right)^2 = (-\xi_1)^2 \tag{3.2}$$

and From Eq. (2.5),

$$\sum_{i=2}^{n} \xi_i^2 = B - \xi_1^2 \tag{3.3}$$

Substitution of Eqs. (3.2) and (3.3) in inequality (3.1) generates the desired result.

□

Theorem 3.2. *For a regular graph* $G \ncong K_2$,

$$\sqrt{B} < LE_{KCD_1}(G) < \sqrt{nB}.$$

Proof. Let $a_i = 1$ and $b_i = |\xi_i|$ in inequality (1.2), generates

$$\left(\sum_{i=1}^{n} |\xi_i|\right)^2 < n\sum_{i=1}^{n} |\xi_i|^2 \tag{3.4}$$

By using Eqs. (2.1)and (2.5) in inequality (3.4), we get

$$LE_{KCD_1}(G) < \sqrt{nB} \tag{3.5}$$

Further, since

$$\left(\sum_{i=1}^{n} |\xi_i|\right)^2 > \sum_{i=1}^{n} |\xi_i|^2$$

Again using Eqs. (2.1)and (2.5) in above inequality, we get

$$LE_{KCD_1}(G) > \sqrt{B} \tag{3.6}$$

Combining Eqs. (3.5) and (3.6), gives the required result.

□

Theorem 3.3. *For a graph* G*,*

$$LE_{KCD_1}(G) \geqslant 2m\left(\frac{n+1}{n}\right). \tag{3.7}$$

Proof. Consider

$$\begin{aligned} LE_{KCD_1}(G) &= \sum_{i=1}^{n} \left|\eta_i - \overline{d}\right| \\ &\geqslant \sum_{i=1}^{n} \eta_i + \overline{d} \end{aligned}$$

With the help of Eq. (2.2), we arrive at

$$LE_{KCD_1}(G) \geqslant 2m\left(\frac{n+1}{n}\right).$$

Equality for inequality (3.7) holds when $G = K_2$. □

Theorem 3.4. *For* $n \geqslant 3$ *and regular graph* $G \ncong K_n$

$$LE_{KCD_1}(G) > \frac{n\eta_1\eta_n + B}{\eta_1 + \eta_n}.$$

Proof. Let

$$\left|\eta_i - \overline{d}\right| = Y_i = |\xi_i|.$$

Consider

$$\sum_{i=1}^{n} (\eta_1 - Y_i)(\eta_n - Y_i) = \sum_{i=1}^{n} \eta_1\eta_n - \sum_{i=1}^{n} (\eta_1 + \eta_n) Y_i + \sum_{i=1}^{n} Y_i^2 < 0.$$

This implies

$$\eta_1\eta_n \sum_{i=1}^{n} 1 - (\eta_1 + \eta_n) \sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} Y_i^2 < 0.$$

With the help of Eqs. (2.1) and (2.5), we arrive at

$$n\eta_1\eta_n - (\eta_1 + \eta_n)\, LE_{KCD_1}(G) + B < 0.$$

On rearranging, we obtain the required result.

□

Theorem 3.5. *For a regular graph* G*,*

$$LE_{KCD_1}(G) = \eta_1 - \eta_n + \sum_{i=2}^{n-1} |\eta_i - \overline{d}| \tag{3.8}$$

Proof. As $\eta_1 > 0$ and $\eta_n < 0$, by Definition (2.1),

$$\begin{aligned} LE_{KCD_1}(G) &= \sum_{i=1}^{n} |\eta_i - \overline{d}| \\ &= |\eta_1 - \overline{d}| + |\eta_n - \overline{d}| + \sum_{i=2}^{n-1} |\eta_i - \overline{d}| \\ &= \eta_1 - \eta_n + \sum_{i=2}^{n-1} |\eta_i - \overline{d}|. \end{aligned}$$

□

Proposition 3.6. *For a regular graph* G*,*

$$LE_{KCD_1}(G) > 2\overline{d}$$

Proof. Consider Eq. (3.8),

$$\begin{aligned} LE_{KCD_1}(G) &= \eta_1 - \eta_n + \sum_{i=2}^{n-1} |\eta_i - \overline{d}| \\ &= \eta_1 - \eta_n + \left| \sum_{i=2}^{n-1} \eta_i - \overline{d} \right| \\ &= \eta_1 - \eta_n + |2m - (\eta_1 + \eta_n) - (n-2)\overline{d}| \\ &= \eta_1 - \eta_n + |2\overline{d} - (\eta_1 - \eta_n)| \\ &= \eta_1 - \eta_n + 2\overline{d} - (\eta_1 - \eta_n) \\ &= 2\left(\overline{d} - \eta_n\right) > 2\overline{d} \end{aligned}$$

□

4. Conclusion

In this article, first KCD Laplacian energy is brought forward as a contribution towards the energy concept in graph theory. Particularly, we define first KCD Laplacian matrix and corresponding first KCD Laplacian energy. Further, we notice that for regular graphs there is equality relationship between first KCD Laplacian energy and first KCD energy. To conclude, an attempt has been made to provide few bounds for $LE_{KCD_1}(G)$

Acknowledgment

Authors are thankful to Karnatak University, Dharwad, Karnataka, India for the support through University Research Studentship (URS), No.KU/Sch/URS/2020-21/1103 dated: 21/12/2020.

References

[1] R. Balakrishnan, *The energy of a graph*, Linear Algebra Appl., **387** (2004), 287–295. 1

[2] S. Bernard, J. M. Child, *Higher Algebra*, Macmillan India Ltd., New Delhi, (2001). 1

[3] I. Gutman, *The energy of a graph, Ber. Math. Stat. Sekt. Forschungsz. Graz*, **103** (1978), 1–22. 1

[4] I. Gutman, B. Zhou, *Laplacian energy of a graph*, Linear Algebra Appl., **414** (2006), 29–37. 1

[5] F. Harary, *Graph Theory*, Addison – Wesely, Reading, Mass, (1969). 1

[6] G. Indulal, I. Gutman, A. Vijaykumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem., **60** (2008), 461–472. 1

[7] K. G. Mirajkar, A. Morajkar, *First* KCD *matrix and first* KCD *energy of a graph*, Accepted for publication in *Mathematical Forum*, **28** (2020). 1

[8] B. Zhou, I. Gutman, T. Aleksic, *A note on Laplacian energy of graphs*, MATCH Commun. Math. Comput. Chem., **60** (2008), 441–446. 1

[9] B. Zhou, *More on energy and Laplacian energy*, MATCH Commun. Math. Comput. Chem., **64** (2010), 75–84. 1